Continue

Set Theory is a branch of mathematics that investigates sets and their properties. The basic concepts of set theory are fairly easy to understand and appear to be self-evident. However, despite its apparent simplicity, set theory turns out to be a very sophisticated subject. In particular, mathematicians have shown that virtually all mathematical concepts and results can be formalized within the theory of sets. This is considered to be one of the greatest achievements of modern mathematics. Given this achievement, one can claim that set theory provides a foundation for mathematics.The foundational role of set theory and its mathematical development have raised many philosophical questions that have been debated since its inception in the late nineteenth century. For example, here are three: Does infinity exist, and if so, are there different kinds of infinity? Is there a mathematical universe? Are all mathematical problems solvable?Before pursuing the philosophical issues concerning set theory, one should be familiar with a standard mathematical development of set theory. This article presents such a development.In the late nineteenth century, the mathematician Georg Cantor (18451918) created and developed a mathematical theory of sets. This theory emerged from his proof of an important theorem in real analysis. In this proof, Cantor introduced a process for forming sets of real numbers that involved an infinite iteration of the limit operation. Cantors novel proof led him to a deeper investigation of sets of real numbers and to his theory of abstract sets. Cantors creation now pervades all of mathematics and offers a versatile tool for exploring concepts that were once considered to be ineffable, such as infinity and infinite sets.Sections 1 and 2 below describe the nave principles of set theory that were used and developed by Cantor. Then, Section 3 describes a more sophisticated (axiomatic) approach to set theory that arose from the discovery of Russells paradox. After identifying the Zermelo-Frankel axioms of set theory, Section 4 discusses Cantors well-ordering principle and examines how Cantor used the well-ordering principle to develop the ordinal and cardinal numbers. Section 5 considers controversies concerning the well-ordering principle and its equivalent, the axiom of choice. This is followed by introducing the cumulative hierarchy of sets, Kurt Gdels universe of constructible sets, and Paul Cohens method of forcing in Sections 6, 7, and 8, respectively. The latter two topics, explored in Sections 7 and 8, can be used to show that certain questions are unresolvable when assuming the Zermelo-Frankel axioms (with or without the axiom of choice). The next two sections address further developments in set theory that are intended to settle these and other unresolved questions; namely, Section 9 discusses large cardinal axioms, and Section 10 investigates the axiom of determinacy.Table of Contents 1. On the OriginsLet us first discuss a few basic concepts of set theory. A set is a well-defined collection of objects. The items in such a collection are called the elements or members of the set. The symbol \(\in\) is used to indicate membership in a set. Thus, if \(A\) is a set, we write \(x \in A\) to say that \(x\) is an element of \(A\), or \(x\) is in \(A\), or \(x\) is a member of \(A\). We also write \(x otin A\) to say that \(x\) is not in \(A\). In mathematics, a set is usually a collection of mathematical objects, for example, numbers, functions, or other sets.Sometimes a set is identified by enclosing a list of its elements by curly brackets; for example, a set of natural numbers \(A\) can be identified by the notation\(A = \{1,2,3,4,5,6,7,8,9\}\).More typically, one forms a set by enclosing a particular expression within curly brackets, where the expression identifies the elements of the set. To illustrate this method of identifying a set, we can form a set B of even natural numbers, using the above set \(A\), as follows:\(B = \{n \in A : n \text{ is even}\}\).which can be read as the set of \(n \in A\) such that \(n\) is even. Of course,\(\{n \in A : n\) is even\(\} = \{2,4,6,8\}\).It is difficult to identify the genesis of the set concept. Yet, the idea of a finite collection of objects has existed for as long as the concept of counting. Indeed, mathematicians have been investigating finite sets and methods for measuring the size of finite sets since the beginning of mathematics. For example, the above two sets\(A=\{1,2,3,4,5,6,7,8,9\}\) are finite sets. As every element in \(B\) is an element in \(A\), the set \(B\) is said to be a subset of \(A\), denoted by \(B \subseteq A\). Since there are elements in \(A\) that are not in \(B\), we say that \(B\) is a proper subset of \(A\). Moreover, the number of elements in \(B\) is strictly smaller than the number of elements in \(A\). Thus, one can say, the whole \(A\) is greater in size than its proper part \(B\).Infinite sets lead to an apparent contradiction. Consider the infinite sets:\(C=\{0,1,2,3,\ldots \}\)\(D=\{1,3,5,7, \ldots \}\).We view the sets \(C\) and \(D\) as existing entities that both contain infinitely many elements. Thus, \(C\) and \(D\) are completed infinities. Observe that every element in \(D\) is in \(C\), and that \(D\) is a proper subset of \(C\). However, if, as many mathematicians once believed, infinity cannot be greater than infinity, then the whole \(C\) is not greater in size than its proper part \(D\). This counterintuitive result was viewed by many early prominent mathematicians as being contradictory, as it appeared to conflict with the well-understood behavior of finite sets. These mathematicians thus concluded that the concept of a completed infinity should not be allowed in mathematics.For this reason, before Cantor, a majority of mathematicians considered infinite collections to be mathematically illicit objects. Cantor was the first mathematician to view infinite sets as being legitimate mathematical objects that can coexist with finite sets. Clearly, the size of a finite set can be measured simply by counting the number of elements in the set. Cantor was the first to investigate the following question:Can the concept of size be extended to infinite sets?Cantor addressed this question in the affirmative by using the concept of a function to measure and compare the sizes of infinite sets. Functions are widely used in science and mathematics. For sets \(A\) and \(B\), we say that \(f\) is a function from \(A\) to \(B\), denoted by \(f\): \(A \rightarrow B\), if and only if \(f\) is a relation (operation) that assigns to each element \(x\) in \(A\), a single element \(f(x)\) in \(B\). There are three important properties that a function might possess:\(f\): \(A \rightarrow B\) is an injection if and only if for each \(y\) in \(B\) there is at most one \(x\) in \(A\) such that \(f(x)=y\).\(f\): \(A \rightarrow B\) is a surjection if and only if for each \(y\) in \(B\) there is at least one \(x\) in \(A\) such that \(f(x)=y\).\(f\): \(A \rightarrow B\) is a bijection if and only if for each \(y\) in \(B\) there is exactly one \(x\) in \(A\) such that \(f(x)=y\).Observe that \(f\): \(A \rightarrow B\) is an injection if and only if distinct elements in \(A\) are assigned to distinct elements in \(B\); that is, for all \(x\) and \(a\) in \(A\), if \(x eq a\), then \(f(x) eq f(a)\). Also note that \(f\): \(A \rightarrow B\) is a bijection if and only if \(f\): \(A\rightarrow B\) is an injection and a surjection.Cantor observed that two sets \(A\) and \(B\) have the same size if and only if there is a one-to-one correspondence between \(A\) and \(B\), that is, there is a way of evenly matching the elements in \(A\) with the elements in \(B\). In other words, Cantor noted that the sets \(A\) and \(B\) have the same size if and only if there is a bijection \(f\): \(A \rightarrow B\). In this case, Cantor said that \(A\) and \(B\) have the same cardinality. For an illustration, let \(\mathbb{N} = \{0, 1, 2, 3, 4, \ldots \}\) be the set of natural numbers and let \(E = \{0,2,4,6,8,\ldots\}\) be the set of even natural numbers. Now let \(f\): \(\mathbb{N} \rightarrow E\) be defined by \(f(n)=2n\). One can verify that \(f\): \(\mathbb{N} \rightarrow E\) is a bijection and, thus, we obtain the following one-to-one correspondence between the set \(\mathbb{N}\) of natural numbers and the set \(E\) of even natural numbers:Hence, each natural number \(n\) corresponds to the even number \(2n\), and each even natural number \(2i\) is thereby matched with \(i \in \mathbb{N}\). The bijection \(f\): \(\mathbb{N} \rightarrow E\) specifies a one-to-one match-up between the elements in \(\mathbb{N}\) and the elements in \(E\). Cantor concluded that the sets N and E have the same cardinality.Cantor also defined what it means for a set \(C\) to be smaller, in size, than a set \(D\). Specifically, he said that \(C\) has smaller cardinality (smaller size) than \(D\) if and only if there is an injection \(f\): \(C \rightarrow D\) but there is no bijection \(g\): \(C \rightarrow D\). Cantor then proved that there is no one-to-one correspondence between the set of real numbers and the set of natural numbers. Cantors proof showed that the set of real numbers has larger cardinality than the set of natural numbers (Cantor 1874). This stunning result is the basis upon which set theory became a branch of mathematics.The natural numbers \(0, 1, 2, 3, \ldots\) are the whole numbers that are typically used for counting. The real numbers are those numbers that appear on the number line. For example, the natural number \(2\), the integer \(-3\), the fraction \(6/5\), and all of the other rational numbers are real numbers. The irrational numbers, such as \(\sqrt{2}\) and \(\pi\), are also real numbers. Again, let \(\mathbb{N} = \{0, 1, 2, 3, \ldots \}\) be the set of natural numbers, and let \(\mathbb{R}\) be the set of real numbers. If a set is either finite or has the same cardinality as the set of natural numbers, then Cantor said that it is countable. Since the set of real numbers \(\mathbb{R}\) is larger, in size, than the set of natural numbers \(\mathbb{N}\), Cantor referred to the set \(\mathbb{R}\) as being uncountable.After proving that the set of real numbers is uncountable, Cantor was able to prove that there is an increasing sequence of larger and larger infinite sets. In other words, Cantor showed that there are infinitely many different infinites, a result with clear philosophical and mathematical significance.After his introduction of uncountable sets, in 1878, Cantor announced his Continuum Hypothesis (CH), which states that every infinite set of real numbers is either the same size as the set of natural numbers or the same size as the entire set of real numbers. There is no intermediate size. Cantor struggled, without success, for most of his career to resolve the Continuum Hypothesis. The problem persisted and became one of the most important unsolved problems of the twentieth century. After Cantors death, most set theorists came to believe that the Continuum Hypothesis is unresolvable.Cantors profound results on the theory of infinite sets were counterintuitive to many of his contemporaries. Moreover, Cantors set theory violated the prevailing dogma that the notion of a completed infinity should not be allowed in mathematics. Thus, the outcry of opposition persisted. Influential mathematicians continued to argue that Cantors work was subversive to the true nature of mathematics. These mathematicians believed that infinite sets were dangerous fictional creations of Cantors imagination and that Cantors fictions needed to be eradicated from mathematics (Dauben 1979, page 1) (Dunham 1990, pp. 278-280). Nevertheless, Cantors theory of sets soon became a crucial tool used in the discovery and establishment of new mathematical results, for example, in measure theory and the theory of functions (Kanamori 2012). Mathematicians slowly began to see the utility of set theory to traditional mathematics. Accordingly, attitudes started to change and Cantors ideas began to gain acceptance in the mathematical community (Dauben 1979, pp. 247-248). The significance of Cantors mathematical research was eventually recognized. David Hilbert, a prominent twentieth century mathematician, described Cantors work as beingthe finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity. (Hilbert 1923)Ultimately, Cantors theory of abstract sets would dramatically change the course of mathematics.2. Cantors Development of Set TheoryIn his development of set theory, Cantor identified a single fundamental principle, called the Comprehension Principle, under which one can form a set. Cantors principle states that, given any specific property \(\varphi(x)\) concerning a variable \(x\), the collection \(\{x : \varphi(x)\}\) is a set, where \(\{x : \varphi(x)\}\) is the set of all objects \(x\) that satisfy the property \(\varphi(x)\). For example, let \(\psi(x)\) be the property that \(x\) is an odd natural number. The Comprehension Principle implies that\(S = \{ x : \psi (x)\} = \{1,3,5,7,\ldots \}\)is a set. Employing the Comprehension Principle, one can form the intersection of two sets \(A\) and \(B\) using the property \(x \in A\) and \(x \in B\); thus, the intersection of \(A\) and \(B\) is the set\(A \cap B = \{x : x \in A\) and \(x \in B\}\).One can also form the set\(A \cup B = \{x : x \in A\) or \(x \in B\}\)which is called the union of \(A\) and \(B\). Recall that one writes \(X \subseteq A\) to mean that \(X\) is a subset of \(A\), that is, every element of \(X\) is also an element of \(A\). Using the Comprehension Principle, one can form the power set of \(A\), which is the set whose elements are all of the subsets of \(A\), that is,\(\wp(A) = \{ X : X \subseteq A\}.\)Thus, if \(A\) is a set and \(X \subseteq A\), then \(X \in \wp(A)\). So, if \(A = \{1,2,3\}\) and \(B = \{3,4,5\}\), then \(A \cup B = \{1,2,3,4,5\}\), and\(\wp(A) = \{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}\),where \(\varnothing\) denotes the empty set, that is, the set that contains no elements. The Comprehension Principle was an essential tool that allowed Cantor to form many important sets. Cantors approach to set theory is often referred to as nave set theory.Cantors set theory soon became a very powerful tool in mathematics. In the early 1900s, the mathematicians mile Borel, Ren-Loius Baire, and Henri Lebesgue used Cantors set theoretic concepts to develop modern measure theory and function theory (Kanamori 2012). This work clearly demonstrated the great mathematical utility of set theory.a. Russells ParadoxThe philosopher and mathematician Bertrand Russell was interested in Cantors work and, in particular, Cantors proof of the following theorem, which implies that the cardinality of the power set of a set is larger than the cardinality of the set. First, recall that a function \(g\): \(A \rightarrow B\) is a surjection (or is onto \(B\)) if for all \(y \in B\), there is an \(x \in A\) such that \(g(x)=y\).Cantors Theorem. Let \(A\) be a set. Then there is no surjection \(f\): \(A \rightarrow \wp(A)\).Proof. Suppose, for the sake of obtaining a contradiction, that there exists a surjection \(f\): \(A \rightarrow \wp(A)\). Observe that, for all \(z \in A\), \(f(z) \subseteq A\). By the Comprehension Principle, let \(X\) be the set\(X = \{x : x \in A\) and \(x otin f(x)\}\).Clearly, \(X \subseteq A\). Thus, \(X \in \wp (A)\). As \(f\) is onto \(\wp(A)\), there is an \(a \in A\) such that \(f(a) = X\). There are two cases to consider: either \(a \in X\) or \(a otin X\). If \(a \in X\), then the definition of \(X\) implies that \(a otin f(a)\). Since \(f(a) = X\), we have that \(a otin X\), which is a contradiction. On the other hand, if \(a otin X\), then the definition of \(X\) implies that \(a \in f(a)\). Since \(f(a) = X\), we see that \(a \in X\), a contradiction. Thus, there is no surjection \(f\): \(A \rightarrow \wp(A)\). This completes the proof.In 1901, after reading Cantors proof of the above theorem, that was published in 1891, Bertrand Russell discovered a devastating contradiction that follows from the Comprehension Principle. This contradiction is known as Russells Paradox. Consider the property \(x otin x\), where \(x\) represents an arbitrary set. By the Comprehension Principle, we conclude that\(A = \{x : x otin x\}\)is a set. The set \(A\) consists of all the sets \(x\) that satisfy \(x otin x\). Clearly, either \(A \in A\) or \(A otin A\). Suppose \(A \in A\). Then, the definition of the set \(A\) implies that \(A\) must satisfy the property \(A otin A\), which contradicts our supposition. Suppose \(A otin A\). Since \(A\) satisfies \(A otin A\), we infer, from the definition of \(A\), that \(A \in A\), which is also a contradiction.There were similar paradoxes discovered by others, including Cantor (Dauben 1979), but Russells paradox is the easiest to understand. These paradoxes appeared to threaten Cantors fundamental principle that he used to develop set theory. Nevertheless, Cantor did not believe that these paradoxes actually refuted his development of set theory. He knew that the construction of certain collections can lead to a contradiction. Cantor referred to these collections as inconsistent multiplicities. Today, such collections are called proper classes, and the paradoxes can be used to prove that they are not sets.3. The Zermelo-Fraenkel AxiomsOver time, it became clear that, to resolve the paradoxes in Cantors set theory, the Comprehension Principle needed to be modified. Thus, the following question needed to be addressed:How can one correctly construct a set?Ernst Zermelo (18711953) observed that to eliminate the paradoxes, the Comprehension Principle could be restricted as follows: Given any set \(A\) and any property \(\psi (x)\), one can form the set \(\{x \in A : \psi (x)\}\), that is, the collection of all elements \(x \in A\) that satisfy \(\psi (x)\), is a set. Zermelos approach differs from Cantors method of forming a set. Cantor declared that for every property one can form a set of all the objects that satisfy the property. Zermelo adopted a different approach: To form a set, one must use a property together with a set.Zermelo also realized that in order to more fully develop Cantors set theory, one would need additional methods for forming sets. Moreover, these additional methods would need to avoid the paradoxes. In 1908, Zermelo published an axiomatic system for set theory that, to the best of our knowledge, avoids the difficulties faced by Cantors development of set theory. In 1930, after receiving some proposed revisions from Abraham Fraenkel, Zermelo presented his final axiomatization of set theory, now known as the Zermelo-Fraenkel axioms and denoted by ZF. These axioms have become the accepted formulation of Cantors ideas about the nature of sets.a. The AxiomsAs noted by Zermelo, to avoid paradoxes, the Comprehension Principle can be replaced with the principle: Given a set \(A\) and a property \(\varphi (x)\) with a variable \(x\), the collection \(\{x \in A : \varphi (x)\}\) is a set. However, this raises a new question: What is a property? The most favored way to address this question is to express the axioms of set theory in the formal language of first-order logic, and then declare that its formulas designate properties. This language involves variables and the logical connectives \(\wedge\) (and), \(\vee\) (or), \(eg\) (not), (if then), and (if and only if), together with the quantifier symbols \(\forall\) (for all) and \(\exists\) (there exists). In addition, this language uses the relation symbols \(=\) and \(\in\) (as well as \(eq\) and \(otin\)). In this language, the variables and quantifiers range over sets and only sets. A formula constructed in this formal language is referred to as a formula in the language of set theory. Such formulas are used to give meaning to the notion of property.We now illustrate the expressive power of this set theoretic language. The formula \(\exists x(x \in A)\) asserts that the set \(A\) is nonempty, and \(\forall x(x otin A)\) states that \(A\) has no elements. Moreover \(eg \exists x \forall y(y \in x)\) states that it is not the case that there is a set that contains all sets as elements. In addition, one can translate English statements, which concern sets, into the language of set theory. For example, the English sentence the set \(A\) contains at least two elements can be translated into the language of set theory by \(\exists x \exists y((x \in A \wedge y \in A) \wedge x eq y)\).There is another quantifier, called the uniqueness quantifier, that is sometimes used. This quantifier is written as \(\exists ! x \varphi (x)\) and it means that there exists a unique \(x\) satisfying \(\varphi (x)\). This is in contrast with \(\exists x \varphi(x)\), which simply states that at least one \(x\) satisfies \(\varphi (x)\). The uniqueness quantifier is used as a convenience, as the assertion \(\exists !x \varphi (x)\) can be expressed in terms of the other quantifiers \(\exists\) and \(\forall\); namely, it is equivalent to the formula\(\exists x \varphi (x) \wedge \forall x \forall y ((\varphi (x) \wedge \varphi (y)) \rightarrow x=y)\).The above formula is equivalent to \(\exists!x \varphi (x)\) because it asserts that there is an \(x\) such that \(\varphi(x)\) holds, and any sets \(x\) and \(y\) that satisfy \(\varphi (x)\) and \(\varphi(y)\) must be the same set.The Zermelo-Fraenkel axioms are listed below. Each axiom is first stated in English and then written in logical form. After each logical form, there is a discussion of the axiom and some of its consequences. When reading these axioms, keep in mind that, in Zermelo-Fraenkel set theory, everything is a set, including the elements of a set. Also, the notation \(\vartheta (x, \ldots, z)\) means that \(x, \ldots, z\) are free variables in the formula \(\vartheta\) and that \(\vartheta\) is allowed to contain parameters (free variables other than \(x, \ldots, z\)) that represent arbitrary sets. Extensionality Axiom. Two sets are equal if and only if they have the same elements.\(\forall A \forall B (A = B \leftrightarrow \forall x (x \in A \leftrightarrow x \in B))\).The extensionality axiom is essentially a definition that states that two sets are equal if and only if they have exactly the same elements. Empty Set Axiom. There is a set with no elements.\(\exists A \forall x (x otin A)\).The empty set axiom states that there is a set which has no elements. Since the extensionality axiom implies that this set is unique, we let \(\varnothing\) denote the empty set. Subset Axiom. Let \(\varphi(x)\) be a formula. For every set \(A\), there is a set \(S\) that consists of all the elements \(x \in A\) such that \(\varphi(x)\) holds.\(\forall A \exists S \forall x (x \in S \leftrightarrow (x \in A \wedge \varphi (x)))\).(The variable \(S\) is assumed not to appear in the formula \(\varphi (x)\).) The subset axiom, also known as the axiom of separation, asserts that any definable sub-collection of a set is itself a set, that is, for any formula \(\varphi(x)\) and any set \(A\), the collection \(\{x \in A : \varphi(x)\}\) is a set. Clearly, the subset axiom is a limited form of the Comprehension Principle. Yet, it does not lead to the contradictions that result from the Comprehension Principle. The subset axiom is, in fact, an axiom schema since it yields infinitely many axioms-one for each formula \(\varphi\). Pairing Axiom. For every \(u\) and \(v\), there is a set that consists of just \(u\) and \(v\).\(\forall u \forall v \exists P \forall x (x \in P \leftrightarrow (x =u \vee x = v))\).The pairing axiom states that, for any two sets \(u\) and \(v\), the set \(\{u, v\}\) exists. Thus, by the extensionality axiom, the set \(\{u, u\} = \{u\}\) exists. Union Axiom. For every set \(F\), there exists a set \(U\) that consists of all the elements that belong to at least one set in \(F\).\(\forall F \exists U \forall x (x \in U \leftrightarrow \exists C (C \in F \wedge x \in C))\).The union axiom states that, for any set \(F\), there is a set \(U\) whose elements are precisely those elements that belong to an element of \(F\), that is, \(x \in U\) if and only if \(x \in A\) for some \(A \in F\). The extensionality axiom implies that the set \(U\) is unique; it is often denoted by \(\bigcup F\). For example, consider the set \(\{A,B\}\). Then\(\bigcup \{A,B\} = \{x : x\) belongs to a member of \(\{A,B\}\} = \{x : x \in A\) or \(x \in B\} = A \cup B\).For another example, let \(F = \{ \{a,b,c\},\{e,f\},\{e,c,d\} \}\). Then \(\bigcup F = \{a,b,c,d,e,f\}\). Power Set Axiom. For every set \(A\), there exists a set \(P\) that consists of all the sets that are subsets of \(A\).\(\forall A \exists P \forall x (x \in P \leftrightarrow \forall y(y \in x \rightarrow y \in A))\).The power set axiom states that, for any set \(A\), there is a set, which we denote by \(\wp(A)\), such that for any set \(B\), \(B \in \wp(A)\) if and only if \(B \subseteq A\). Infinity Axiom. There is a set \(I\) that contains the empty set as an element and whenever \(x \in I\), then \(x \cup \{x\} \in I\).\(\exists I (\varnothing \in I \wedge \forall x (x \in I \rightarrow x \cup \{ x \} \in I))\).The infinity axiom ensures the existence of at least one infinite set. For any set \(x\), the successor of \(x\) is defined to be the set \(x^{+} = x \cup \{x\}\). Thus, the axiom of infinity asserts that there is a set \(I\) such that \(\varnothing \in I\) and if \(x \in I\), then \(x^{+} \in I\). Note that \(\varnothing^{+} = \{\varnothing\}\), and that \(\{\varnothing\}^{+} = \{\varnothing,\{\varnothing\}\}\). It follows that the set \(I\) contains each of the sets\(\varnothing; \{\varnothing\}; \{\varnothing, \{\varnothing \}\}; \{\varnothing, \{\varnothing, \{\varnothing \}\}\}; \ldots\).One can show that any two of the sets in the above list (separated by a semi-colon) are distinct. Hence, the set \(I\) contains an infinite number of elements; in other words, \(I\) is an infinite set. So, the infinity axiom simply states that infinite sets exist and are legitimate mathematical objects. The infinity axiom is a key tool that is used to develop the set of natural numbers \(\mathbb{N}\) and to prove that \(\mathbb{N}\) is well-ordered, that is, every nonempty set of natural numbers has a least element. Replacement Axiom. Let \(\psi (x, y)\) be a formula. For every set \(A\), if for each \(x \in A\) there is a unique \(y\) such that \(\psi (x, y)\), then there is a set \(S\) that consists of all of the elements \(y\) such that \(\psi (x, y)\) for some \(x \in A\). (Below, \(\exists\)! is the uniqueness quantifier.)\(\forall A (\forall x (x \in A \rightarrow \exists ! y \psi (x,y)) \rightarrow \exists S \forall y(y \in S \leftrightarrow \exists x (x \in A \wedge \psi(x, y))))\).(The variable \(S\) is assumed not to appear in the formula \(\psi (x, y)\).) The replacement axiom states that for every set \(A\), if for each \(x \in A\) there is a unique \(y\) such that \(\psi(x,y)\), then the collection \(\{y\} : \(\exists x (x \in A \wedge \psi(x,y))\}\) is a set; that is, a functional image of a set, is a set. The replacement axiom is a special form of Cantors Comprehension Principle that plays a critical role in modern set theory. However, the replacement axiom does not lead to the contradictions that follow from the Comprehension Principle. Like the subset axiom, the replacement axiom is an axiom schema. Accordingly, there are infinitely many Zermelo-Fraenkel axioms. Regularity Axiom. Each nonempty set \(A\) contains an element that is disjoint from \(A\).\(\forall A (A eq \varnothing \rightarrow \exists x (x \in A \wedge eg \exists y (y \in x \wedge y \in A)))\).The regularity axiom, also known as the axiom of foundation, states that, for any nonempty set \(A\), there is a set \(x \in A\) such that \(A \cap x = \varnothing\). The regularity axiom rules out the possibility of a set belonging to itself. In standard mathematics, there are no sets that are members of themselves. For example, the set of natural numbers is not a natural number. The regularity axiom eliminates collections that are not relevant for standard mathematics. The regularity and pairing axioms imply that if \(a \in b\), then \(b otin a\). To see this, suppose that \(a \in b\). Then it follows, from regularity, that \(a \cap \{a,b\} = \varnothing\). So \(b otin a\).The Zermelo-Fraenkel axioms are now the most widely accepted answer to the question: How can one correctly construct a set? Of course, these axioms are more restrictive than Cantors Comprehension Principle; however, no one, in over 100 years, has been able to derive a contradiction from these axioms. Moreover, all of the classic results (excluding the paradoxes) that were derived using Cantors nave set theory can be derived from the Zermelo-Fraenkel axioms.It is a remarkable fact that essentially all mathematical objects can be defined as sets within Zermelo-Fraenkel set theory. For example, functions, relations, the natural numbers, and the real numbers can be defined within Zermelo-Fraenkel set theory. Hence, effectively all theorems of mathematics can be considered as statements about sets and proven from the Zermelo-Fraenkel axioms.b. ClassesThe argument used in Russells Paradox can be applied to prove, in ZF, that there is no set that contains all sets (as elements). As every set is equal to itself, the collection \(\{x\) : \(x = x\}\) contains every set, but this collection is not a set. Thus, given a formula \(\varphi(x)\), one cannot necessarily conclude that the collection \(\{x : \varphi(x)\}\) is a set. However, in set theory, it is convenient to be able to discuss such collections. They cannot be called sets. Instead, a collection of the form \(\{x\} : \(\varphi(x)\}\) is called a class. The collection \(\{x : x = x\}\) is a class that is not a set; for this reason, it is called a proper class.When can one prove that a class is a set? Let us say that a class \(\{x : \varphi(x)\}\) is bounded if and only if there is a set \(A\) such that for all \(x\), if \(\varphi(x)\), then \(x \in A\). Using the subset axiom, one can prove that a bounded class is a set. It follows that the class \(\{x : x = x\}\) is not bounded.In the Zermelo-Fraenkel axioms, there is no explicit mention of classes. However, there are alternative axiomatizations of set theory that extend ZF by including classes as objects in the language, that is, these axiom systems give classes a formal state of existence. The most common such axiomatic treatment of classes is denoted by NBG (von NeumannBernaysGdel). The NBG system uses a formal language that has two different types of variables: capital letters denote classes and lowercase letters denote sets. In addition, classes can contain only sets as elements. So, a class that is not a set cannot belong to a class. Thus, a class \(X\) is a set if and only if \(\exists Y (X \in Y)\). In the NBG system, sets satisfy all of the ZF axioms, and the intersection of a class with a set is a set, that is, \(X \cap y\) is a set. The NBG system also has the class comprehension axiom:\(\exists X \forall y (y \in X \leftrightarrow \varphi (y))\)where the formula \(\varphi(y)\) can contain set parameters and/or class parameters (with other restrictions). Thus, the class comprehension axiom asserts that \(\{x : \varphi(x)\}\) is a class.The NBG system is a conservative extension of ZF; that is, a sentence with only lowercase (set) variables is provable in NBG if and only if it is provable in ZF. The Zermelo-Fraenkel system has a clear advantage over NBG, namely, the simplicity of working with only one type of object (sets) rather than two types of objects (sets and classes). The Zermelo-Fraenkel axiomatic system is the standard system of axioms for modern set theory.4. Cantors Well-Ordering PrincipleAs proposed by Cantor, two sets \(A\) and \(B\) have the same cardinality if and only if there is a bijection \(f\): \(A \rightarrow B\). When \(A\) is a finite set, there is a unique natural number, denoted by |\(A\)|, that identifies the number of elements in \(A\). In this case, we say that |\(A\)| is the cardinality of \(A\). For example, if \(A = \{3,5,7,2\}\), then |\(A\)| \(= 4\). Clearly, the cardinality of a finite set identifies the number of elements that are in the set. Moreover, if \(A\) and \(B\) are both finite sets, then one can prove that|\(A\)| = |\(B\)| if and only if there exists a bijection \(f\): \(A \rightarrow B\). With this understanding, Cantor asked the following question:Are there values that can represent the size of infinite sets and satisfy (\(\Delta\))?In other words, given two infinite sets \(A\) and \(B\), can one assign values |\(A\)| and |\(B\)| such that|\(A\)| = |\(B\)| if and only if there exists a bijection \(f\): \(A \rightarrow B\)?Cantor answered this question, in the affirmative, by developing the transfinite ordinal numbers, which are infinite numbers in the sense that they are larger than all of the natural numbers, and are well-ordered just like the natural numbers. Cantor believed that each infinite set can be assigned a specific ordinal number and that this ordinal number would measure the size of the set. Cantor realized that, in order to successfully apply his theory of ordinal numbers, he needed an additional principle. In 1883, he proposed the following principle.Well-Ordering Principle: It is always possible to bring any well-defined set into the form of a well-ordered set.A relation \(\leq\) on a set \(X\) is a well-ordering of \(X\) if and only if it is a total ordering in which every non-empty subset of \(X\) has a least element, where it is assumed that the relation \(\leq\) does not apply to any elements that are not in \(X\). If a set can be well-ordered, then one can generalize the concepts of induction and recursion, similar to mathematical induction, on the elements of the set. Given any infinite set, Cantor used the well-ordering principle to identify an ordinal number that measures the size of the set. Such an ordinal is called a cardinal number.a. Ordinal NumbersThe natural numbers are often used for two purposes: to indicate the position of an element in a sequence and to identify the size of a finite set. In other words, a natural number can be used to identify a position (first, second, third,) and it can be used to identify a size (one, two, three,). Cantor extended the natural numbers by introducing the concepts of transfinite position and transfinite size. Suppose that we want to count the number of real numbers. As noted in Section 1, Cantor proved that the set of real numbers is uncountable. Thus, if we attempted to assign each real number to exactly one of the natural numbers \(0, 1, 2, 3, \ldots,\) then we would not have enough natural numbers to complete this task. However, suppose that we add some new numbers, called transfinite ordinals, to our stock of numbers. Clearly, we need an ordinal that will identify the first position that occurs after all of the natural numbers. Cantor denoted this ordinal by the Greek letter \(\omega\). That is, Cantor proposed the following position sequence\(0, 1, 2, 3, 4, \ldots, \omega\). Observe the following:By starting with \(0\) and repeatedly adding \(1\), we obtain all of the natural numbers.Every natural number greater than \(0\) has an immediate predecessor; for example, \(5\) has \(4\) as its immediate predecessor.By contrast, the ordinal number \(\omega\) cannot be obtained by repeatedly adding \(1\) to \(0\) and it does not have an immediate predecessor. For these reasons, we say that \(\omega\) is a limit ordinal.We can continue the sequence (1) by repeatedly adding \(\) to \(\omega\). By doing so, we obtain the following position sequence:\(0, 1, 2, 3, 4, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots\) The process for constructing (1) and (2) can be repeated endlessly. In this way, we obtain the ordered sequence of all of the ordinals:\(0, 1, 2, 3, 4, \ldots, \omega, \omega+1, \omega+2, \ldots ,\omega+\omega,(\omega+\omega)+1,(\omega+\omega)+2, \ldots\) where \(\omega+\omega\) is a limit ordinal which is usually represented by \(2 \cdot \omega\). An ordinal of the form \(\alpha+1\) is called a successor ordinal. An ordinal \(\delta\) > \(0\) that is not a successor ordinal is called a limit ordinal. Cantor used the ordinals to measure the length of a well-ordered set.The natural numbers \(0, 1, 2, 3, 4, \ldots\) are sometimes called finite ordinals. Every nonempty subset of the natural numbers has a least element. Similarly, every nonempty set of ordinals has a least element with respect to the ordering in (3). The ordinal numbers are a generalized extension of the natural numbers. One can define the operations of addition, multiplication, and exponentiation on the ordinal numbers. These operations satisfy some (but not all) of the arithmetic properties that hold on the natural numbers, for example, addition is associative (Cunningham 2016).The set of predecessors of an ordinal is the set of all of the ordinals that come before it in the list (3); for example, the set of predecessors of \(\omega\) and \(\omega+1\) are the respective sets\(\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}\), \(N = \{0, 1, 2, 3, 4, \ldots , \omega \}\). The ordinals \(\omega\) and \(\omega+1\) represent different positions in the list (3); but, the sets \(\mathbb{N}\) and \(N\) in (4) have the same cardinality. Note that the cardinality of \(\mathbb{N}\) is larger than any finite set, that is, for any natural number \(n\), the set \(\mathbb{N}\) has cardinality larger than the set \(\{0, 1, 2, \ldots, n\}\). For this reason, we say that \(\omega\) is a cardinal number.For any two ordinals \(\alpha\) and \(\beta\), we say that \(\alpha\) < \(\beta\) if and only if \(\alpha\) appears before \(\beta\) in the list (3). For each ordinal \(\gamma\), let Pred(\(\gamma\)) = \(\{\alpha : \alpha\) < \(\gamma\}\) be the set of predecessors of \(\gamma\). One can prove, in ZF, that Pred(\(\gamma\)) is a set. In contemporary set theory one usually defines the ordinals so that, for each ordinal \(\gamma\), \(\gamma\) = Pred(\(\gamma\)); that is, each ordinal is defined to be the set of its predecessors. Specifically, a set \(\gamma\) is said to be an ordinal if and only if \(\gamma\) is well-ordered by the membership relation and is transitive, that is, every element in \(\gamma\) is a subset of \(\gamma\). Thus, if \(\alpha\) < \(\beta\), then \(\alpha \in \beta\) and \(\alpha \subseteq \beta\). For example, \(\omega = \{0, 1, 2, 3, 4, \ldots\}\) is an ordinal if the integers (the finite ordinals) are defined as follows:\(0 = \varnothing\),\(1 = \{0\}\),\(2 = \{0,1\}\),\(3 = \{0,1,2\}\),\(4 = \{0,1,2,3\}\).This approach is due to Von Neumann (2009), and such ordinals can be called Von Neumann ordinals. The collection of all ordinals is a proper class (see Cunningham 2016).b. Cardinal NumbersAn ordinal number \(\kappa\) is said to be a cardinal if and only if, for all \(\alpha\) < \(\kappa\), the set Pred(\(\alpha\)) has smaller cardinality than Pred(\(\kappa\)). It follows that the natural numbers are all cardinals. As noted above, \(\omega\) is the first transfinite cardinal, which is often denoted by \(\aleph_{0}\). The next transfinite cardinal, after \(\aleph_{0}\), is designated by \(\aleph_{1}\). This process can be continued to produce the following sequence of finite and transfinite cardinals:\(0, 1, 2, 3, 4, \ldots, \aleph_{0}, \aleph_{1}, \ldots, \aleph_{\omega}, \aleph_{\omega+1}, \ldots, \aleph_{2 \cdot \omega}, \ldots, \aleph_{\omega \cdot \omega}, \ldots\) where the transfinite cardinal numbers in (5) are indexed by the ordinal numbers. Thus, the collection of all the cardinal numbers is a proper class. A cardinal \(\aleph_{\beta}\) is called a successor cardinal if and only if \(\beta\) is a successor ordinal; otherwise, it is called a limit cardinal. One can prove, in ZF, that, for every cardinal \(\kappa\), there is an ordinal \(\alpha\) such that \(\kappa = \aleph_{\alpha}\) (Cunningham 2016). Thus, every cardinal appears on the list (5). One can define the operations of addition, multiplication, and exponentiation on the cardinals (exponentiation requires the well-ordering principle). These particular operations are not the same as the corresponding operations on the ordinal numbers (Cunningham 2016).Cantor used the cardinal numbers to measure the size of sets. The well-ordering principle implies that every set A can be assigned a (unique) cardinal number that measures its size. This cardinal number is usually denoted by |\(A\)|, and is called the cardinality of \(A\). Cantors Theorem implies that, for any set \(A\), |\(A\)| < |\(\wp(A)\)|. The operation of cardinal exponentiation allowed Cantor to prove that the cardinality of \(\mathbb{R}\), the set of real numbers, is equal to \(2^{\aleph_{0} }\), that is, |\(\mathbb{R}\)| = \(2^{\aleph_{0}}\). Since \(\aleph_{1}\) is the first cardinal greater than \(\aleph_{0}\), Cantor was able to express the Continuum Hypothesis in terms of the equation \(2^{\aleph_{0}} = \aleph_{1}\). Moreover, assuming the well-ordering principle, one can conclude that a set \(A\) is countable if and only if |\(A\)| \(leq \aleph_{0}\) and that a set \(B\) is u

mathematics can be viewed as assertions about sets. Moreover, these theorems can also be proven from ZFC, the Zermelo-Fraenkel axioms together with the axiom of choice. Cohens forcing method clearly shows that ZFC is an incomplete theory, as there are statements that cannot be resolved in it. This motivates the following question:What path should be taken to try to settle the Continuum Hypothesis and other undecided statements in mathematics?In contemporary set theory, the most common answer to this question is called Gdels Program:Search for new axioms, which, when added to ZFC, will determine the truth or falsity of unresolved statements.This program was inspired by an article of Gdels in which he discusses the mathematical and philosophical aspects of mathematical statements that are independent of ZFC (Gdel 1947). Sections 9 and 10 will discuss two directions that this program has taken: large cardinal axioms and determinacy axioms.9. Large Cardinal AxiomsRoughly, a large cardinal axiom is a set-theoretic statement that asserts the existence of an uncountable cardinal \(\kappa\) that satisfies a particular property that implies that there is a set \(M\) such that \((M,\in)\) is a model of ZFC; such a \(\kappa\) is called a large cardinal. Gdels second incompleteness theorem implies that, in ZFC, one cannot prove the existence of large cardinals. Thus, a large cardinal axiom is a new axiom. Most modern set theorists believe that the standard large cardinal axioms are consistent with ZFC.Assuming ZFC, let us say that a cardinal \(\kappa\) is a strong limit cardinal if and only if, for every cardinal \(\lambda\), if \(\lambda\) < \(\kappa\), then \(2^{\lambda}\) < \(\kappa\). A cardinal \(\kappa\) is said to be inaccessible if and only if \(\kappa\) is uncountable, regular, and a strong limit cardinal. Recall that a cardinal \(\kappa\) is regular if \(\kappa\) is not the union of fewer than \(\kappa\) many sets of size each less than \(\kappa\). If \(\kappa\) is an inaccessible cardinal, then, in ZFC, one can prove that \((V_{\kappa},\in)\) is a model of ZFC (Kanamori 2003). Hence, such a \(\kappa\) is an example of a large cardinal and so, the statement there exists an inaccessible cardinal is a large cardinal axiom.There are other large cardinal axioms. The description of these large cardinal axioms usually involves the concept of an elementary embedding of the universe, that is, a nontrivial truth preserving transformation from \((V,\in)\) into \((M,\in)\) where \(M\) is a transitive subclass of \(V\). A theorem of Kenneth Kunen (Jech 2003) shows that there is no nontrivial elementary embedding of the universe \(V\) into itself. Thus, for any nontrivial truth preserving transformation from \((V,\in)\) into \((M,\in)\) where \(M\) is a transitive subclass of \(V\), \(M eq V\). More specifically, a large cardinal axiom can be expressed as asserting that there exists a nontrivial (class) function such that for each formula \(\varphi(v_{1},v_{2},\ldots,v_{n})\) (in the language of set theory) and for all elements \(x_{1},\ldots,x_{n}\) in \(V\),\((V,\in) \vDash \varphi(x_{1},\ldots,x_{n})\) if and only if \((M,\in) \vDash \varphi(j(x_{1}),\ldots,j(x_{n})))\.Since the embedding \(j\) is not the identity, there must be a least ordinal \(\kappa\) such that \(\kappa\) < \(j(\kappa)\). This ordinal is called the critical point of \(j\) and is denoted by \(\kappa\) = crit\((j)\). It follows that \(\kappa\) is a cardinal; indeed, \(\kappa\) is the large cardinal that is confirmed by the existence of the embedding \(j\).A cardinal \(\kappa\) is said to be measurable if and only if there exists an embedding \(j: V \rightarrow M\) such that \(\kappa\) is the critical point of \(j\). In this case, one can prove that \(V_{\kappa+1} \subseteq M\). Therefore, there is some resemblance between \(M\) and \(V\). Increasingly stronger large cardinal axioms demand a greater agreement between \(M\) and \(V\). For example, if one requires that \(V_{\kappa+2} \subseteq M\), then one obtains a stronger large cardinal axiom. For another example, a cardinal \(\kappa\) is said to be superstrong if and only if there is a transitive class \(M\) and a nontrivial elementary embedding \(j: V \rightarrow M\) such that \(\kappa\) = crit\((j)\) and \(V_{j(\kappa)} \subseteq M\). Even stronger large cardinal axioms are obtained by requiring greater and greater resemblance between \(M\) and \(V\) (Woodin 2011).Large cardinal axioms are statements that assert the existence of large cardinals. These axioms are widely viewed as being very promising new axioms for set theory. Large cardinal axioms do not resolve the Continuum Hypothesis but they have led mathematicians to formulate conditions under which Cantors hypothesis is false (Woodin 2001, p. 688). As already mentioned, one cannot prove, in ZFC, that large cardinals exist. Yet, there is very strong evidence that their existence cannot be refuted in ZFC (Maddy 1988).10. The Axiom of DeterminacyDescriptive set theory has its origins, in the early 20th century, with the theory of real-valued functions and sets of real numbers developed by Borel, Baire, and Lebesgue. These analysts, respectively, introducedthe hierarchy of Borel sets of real numbers,the Baire hierarchy of real-valued functions,Lebesgue measurable sets of real numbers.Descriptive set theory extends the work of these mathematicians (Moschovakis 2009). Recall that \(\omega = \{0,1,2,3,4,\ldots\}\) is the set of natural numbers. Let \(^{\omega}\omega\) be the set of all functions from \(\omega\) to \(\omega\). The set \(^{\omega}\omega\) is denoted by \(\mathbb{R}\) and is called Baire Space. \(\mathbb{R}\) is often referred to the set of reals; and if \(x \in \mathbb{R}\), then \(x\) is called a real. \(\mathbb{R}\) is regarded as a topological space by giving it the product topology, using the discrete topology on \(\omega\). The space \(\mathbb{R}\) is homeomorphic to the set of irrational numbers which is a subspace of the set of real numbers (Moschovakis 2009).Descriptive set theory is a branch of set theory that uses set theoretic tools to investigate the structure of definable sets and functions over \(\mathbb{R}\). One can identify the level of complexity of such definable sets of reals (Moschovakis 2009). Thus, there is a natural hierarchy on the definable subsets of \(\mathbb{R}\), which, in increasing order of complexity, is called the projective hierarchy.As a result of Gdels and Cohens work, it has been shown that many questions in descriptive set theory are not decidable in axiomatic set theory. For example, in 1938, Gdel showed that in \(L\), the universe of constructible sets, there are projective sets of reals that are not Lebesgue measurable. In 1970, using the method of forcing, Robert Solovay showed that if there is an inaccessible cardinal, then ZFC is consistent with the statement that every projective set is Lebesgue measurable. Thus, one can neither prove nor disprove, in ZFC, the Lebesgue measurability of projective sets. Hence, in ZFC, the theory of projective sets is incomplete. For this reason, modern descriptive set theory focuses on new axioms; one such axiom concerns infinite games.Gale and Stewart (1953) introduced the general concept of an infinite game of perfect information and began the study of these games. Other mathematicians then pursued this subject and discovered that it can be used to resolve problems in descriptive set theory.We now turn to a description of infinite games and strategies. For each \(A \subseteq \mathbb{R}\), we associate a two-person infinite game on \(\omega\) with payoff \(A\), denoted by \(G_{A}\), where players I and II alternately choose natural numbers \(a_{i}\) in the order given in the diagram:After completing an infinite number of moves, the players produce the real\(x =\) \(a_{0},a_{1},a_{2},\ldots\).Player I is said to win if \(x \in A\), otherwise player II is said to win. As each player is aware of all the previous moves before making a next move, the game is called a game of perfect information. The game \(G_{A}\) is said to be determined if and only if either player has a winning strategy, that is, a function that ensures the player will win the game regardless of how the other player makes his or her moves. The Axiom of Determinacy (AD) is a regularity hypothesis about such games that states: For all \(A \subseteq \mathbb{R}\), the game \(G_{A}\) is determined.In the theory ZF+AD, one can resolve many open questions about the sets of real numbers. For example, one can prove Cantors original form of the continuum hypothesis: Every uncountable set of real numbers has the same cardinality as the full set of real numbers.Moreover, it has been shown that the axiom of choice implies that AD is false; that is, using the axiom of choice, one can construct a set of reals \(A\) such that the game \(G_{A}\) is not determined. Thus, the axiom of determinacy is incompatible with the axiom of choice. However, it is not clear that one can establish, without the axiom of choice, the existence of a set of reals \(A\) such that the game \(G_{A}\) is not determined (Moschovakis 2009). Moreover, there are weaker versions of AD that are compatible with ZF together with a weaker choice principle called the axiom of dependent choices.Axiom of Dependent Choices (DC). Let \(R\) be a relation on a nonempty set \(A\). Suppose that for all \(x \in A\) there is a \(y \in A\) such that \(R(x,y)\). Then there exists a function \(f: \omega \rightarrow A\) such that, for all \(n \in \omega\), \(R(f(n),f(n+1))\).Many mathematicians working in descriptive set theory operate within the background theory ZF+DC and the following determinacy axiom: For every projective set \(A\), the game \(G_{A}\) is determined. This axiom is denoted by PD (projective determinacy). Under the theory ZF+DC+PD, the classic open questions about projective sets have been successfully addressed (Moschovakis 2009). In particular, this theory implies that all projective sets are Lebesgue measurable.Generalizing the construction of the inner model \(L\), one can construct the inner model \(L(\mathbb{R})\), the smallest inner model that contains all the ordinals and all the reals. The set \(wp(\mathbb{R}) \cap L(\mathbb{R})\) can be viewed as a natural extension of the projective sets. The determinacy hypothesis denoted by AD\(^{L(\mathbb{R})}\), asserts that AD holds in \(L(\mathbb{R})\). Since the inner model \(L(\mathbb{R})\) contains all of the projective sets, the assumption AD\(^{L(\mathbb{R})}\) implies PD.There are very deep results that connect determinacy hypotheses and large cardinal axioms. In 1988, Martin and Steel, working in ZFC, identified a large cardinal axiom that implies PD. By assuming a stronger large cardinal axiom, Woodin, within ZFC, was able to prove that AD\(^{L(\mathbb{R})}\) holds and so, \(L(\mathbb{R})\) satisfies ZF+AD. Moreover, PD and AD\(^{L(\mathbb{R})}\), individually, imply the consistency of certain large cardinal axioms (Kanamori 2003). Investigating the relationships between determinacy hypotheses and large cardinals has become an important component of modern set theory.11. Concluding RemarksSet Theory is a rich and beautiful branch of mathematics whose fundamental concepts permeate all branches of mathematics. It is a most extraordinary fact that all standard mathematical objects can be defined as sets. For example, the natural numbers and the real numbers can be constructed within set theory. In addition, algebraic structures, functional spaces, vector spaces, and topological spaces can be viewed as sets in the universe of sets \(V\). Consequently, mathematical theorems can be regarded as statements about sets. These theorems can also be proven from ZFC, the axioms of set theory. Thus, mathematics can be embedded into set theory.Since all of conventional mathematics can be developed within set theory, one can view certain results in set theory as being part of metamathematics, the field of study within mathematics that uses mathematical tools to investigate the nature and power of mathematics. For example, using the forcing technique and inner models, it has been shown that there are mathematical statements that cannot be proven or disproven in ZFC. Thus, when a particular mathematical statement is unresolved, set theory can sometimes show that there is neither a proof nor a refutation of the statement in ZFC. As noted above, this situation has inspired the search for new set theoretic axioms.Of course, the fact that set theory offers a foundation for mathematics indicates that set theory is a very important branch of mathematics. However, the concepts and techniques developed within set theory demonstrate that, in itself, set theory is a deep and exciting branch of mathematics with significant applications to other areas of mathematics. This success has inspired some philosophers of mathematics to direct their attention to the philosophy of set theory and the search for new axioms (Maddy 1988a, 1988b, 2011).12. References and Further Readinga. Primary SourcesBanach, S. and Tarski, A. 1924. Sur la dcomposition des ensembles de points en parties respectivement congruentes, Fund. Math., 6, pp. 244277.Cantor, Georg. 1874. ber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen, Journal fur die reine und angewandte Mathematik (Crelle). 77, 258262.Cohen, Paul J. 1963. The independence of the axiom of choice. Mimeographed.Cohen, Paul J. 1963a. The independence of the continuum hypothesis I. Proceedings of the U.S. National Academy of Sciences 50, 1143-48.Cohen, Paul J. 1964. The independence of the continuum hypothesis II. Proceedings of the U.S. National Academy of Sciences 51, 105-110.Cohen, Paul J. 1966. Set Theory and the Continuum Hypothesis, New York: Benjamin.Cunningham, Daniel W. 2016. Set Theory: A First Course, New York: Cambridge University Press.Dauben, Joseph W. 1979. Georg Cantor: his mathematics and philosophy of the infinite, Cambridge, Mass., Harvard University Press; reprinted: Princeton, Princeton University Press, 1990.Dunham, William. 1990. Journey Through Genius: The Great Theorems of Mathematics (1st ed.). John Wiley and Sons.Gale, D. and Stewart, F.M. 1953. Infinite games with perfect information, Annals of Math. Studies, vol. 28, pp. 245266.Gdel, Kurt. 1947. What is Cantors Continuum Problem?, American Mathematical Monthly, vol. 54, pp. 515-525.Gdel, Kurt. 1986. Collected Works, Volume I: Publications 19291936, (Solomon Feferman, editor-in-chief), Oxford University Press, New York.Gdel, Kurt. 1990. Collected Works, Volume II: Publications 19381974, (Solomon Feferman, editor-in-chief), Oxford University Press, New York.Gdel, Kurt. 1995. Collected Works, Volume III: Unpublished Essays and Lectures, (Solomon Feferman, editor-in-chief), Oxford University Press, New York.Hilbert, David. 1923. On the infinite. Reprinted in the Philosophy of Mathematics: Selected Readings, 1983, edited by Paul Benacerraf and Hilary Putnam, pp. 83-201.Jech, Thomas. 2003. Set theory. Third Edition, New York: Springer.Jech, Thomas. 1973. The Axiom of Choice, North-Holland Publishing Company, Studies in logic and the foundations of mathematics, vol. 75, Amsterdam.Kanamori A. 2003. The Higher Infinite. Perspectives in Mathematical Logic. Second edition. Berlin: Springer.Kanamori A. 2012. Set theory from Cantor to Cohen, a book chapter in: Handbook of the History of Logic: Sets and Extensions in the Twentieth Century. Volume editor: Akihiro Kanamori. General editors: Dov M. Gabbay, Paul Thagard and John Woods. Elsevier BV.Kunen, Kenneth. 2009. The Foundations of Mathematics. Studies in Logic, vol. 19. London: College Publications.Kunen, Kenneth. 2011. Set Theory. Studies in Logic, vol. 34. London: College Publications.Lvy, Azriel. 1960. Axiom schemata of strong infinity in axiomatic set theory, Pacific Journal of Mathematics, 10, pp. 223238.Maddy, Penelope H. 1988a. Believing the axioms I. The Journal of Symbolic Logic, 53(2), 481511.Maddy, Penelope H. 1988b. Believing the axioms II. The Journal of Symbolic Logic, 53(3), 736764.Maddy, Penelope H. 2011. Defending the axioms. On the philosophical foundations of set theory. Oxford: Oxford University Press.Montague, Richard M. 1961. Fraenkels addition to the axioms of Zermelo. Essays on the foundations of mathematics, dedicated to A. A. Fraenkel on his seventieth anniversary, edited by Y. Bar-Hillel, E. I. J. Poznanski, M. O. Rabin, and A. Robinson for The Hebrew University of Jerusalem, Magnes Press, Jerusalem, and North-Holland Publishing Company, Amsterdam, pp. 91114.Moore, Gregory H. 2012. Zermelos Axiom of Choice: Its Origins, Development, and Influence. Mineola, NY: Dover Publications. Reprint of the 1982 original published by Springer.Moschovakis, Yiannis. 2009. Descriptive Set Theory, 2nd edition, vol. 155 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, 2009.Solovay, Robert. 1970. A model of set theory in which every set is Lebesgue measurable. Annals of Mathematics, vol. 92, 156.Shelah, Saharon. 1974. Infinite abelian groups, Whitehead problem and some constructions. Israel J. Math, vol. 18, 243256.Woodin, Hugh. 2001. The Continuum Hypothesis, Part II. Notices of the American Mathematical Society, vol. 48, no. 7.Woodin, Hugh. 2011. Infinity, a book chapter in: Infinity: New Research Frontiers. Cambridge: Cambridge University Press.Zermelo, Ernst. 2010. Collected Works. Gesammelte Werke. Volume I: Set Theory, Miscellania. Mengenlehre, Varia, edited by H.-D. Ebbinghaus and A. Kanamori, Springer, Berlin and Heidelberg, xxiv + 654 pp.b. Secondary SourcesEbbinghaus, Heinz-Dieter. 2007. Ernst Zermelo. An Approach to His Life and Work. Berlin: Springer. In cooperation with Volker Peckhaus.Enderton, Herbert B. 1977. Elements of Set Theory. New York: Academic Press.Enderton, Herbert B. 2001. A Mathematical Introduction to Logic. 2nd edn. Burlington, MA: Harcourt/Academic Press.Feferman, Solomon, Parsons, Charles and Simpson, Steven G. (Eds.). 2010. Kurt Gdel: essays for his centennial. Cambridge: Cambridge University Press.Halmos, Paul R. 1974. Nave Set Theory. New York: Springer. Reprint of the 1960 edition published by Van Nostrand.Hauser, Kai. 2006. Gdels Program Revisited Part I: The Turn to Phenomenology. Bulletin of Symbolic Logic, 12(4), 529590.Heller, Michael and Woodin, Hugh. (Eds.). 2011. Infinity: New Research Frontiers. Cambridge: Cambridge University Press.Kanamori, Akihiro. 2012. In praise of replacement. Bulletin of Symbolic Logic, 18(1), 4690.Levy, Azriel. 2002. Basic Set Theory. Mineola, NY: Dover Publications. Reprint of the 1979 original published by Springer.Moschovakis, Yiannis. 2006. Notes on Set Theory. 2nd edition. Undergraduate Texts in Mathematics. New York: Springer.Potter, Michael. 2004. Set theory and Its Philosophy. New York: Oxford University Press.c. Internet Sources Author InformationDaniel CunninghamEmail: cunnindw@buffalostate.eduState University of New York Buffalo StateU. S. A.Branch of mathematics that studies setsThis article is about the branch of mathematics. For other uses, see Set theory (disambiguation).Not to be confused with Set theory (music).A Venn diagram illustrating the intersection of two setsPart of a series onMathematicsHistoryIndexAreasNumber theoryGeometryAlgebraCalculus and AnalysisDiscrete mathematicsLogicSet theoryProbabilityStatistics and Decision theoryRelationship with sciencesPhysicsChemistryGeosciencesComputationBiologyLinguisticsEconomicsPhilosophyEducation Mathematics PortalvteSet theory is the branch of mathematical logic that studies sets, which can be informally described as collections of objects. Although objects of any kind can be collected into a set, set theory as a branch of mathematics is mostly concerned with those that are relevant to mathematics as a whole.The modern study of set theory was initiated by the German mathematicians Richard Dedekind and Georg Cantor in the 1870s. In particular, Georg Cantor is commonly considered the founder of set theory. The non-formalized systems investigated during this early stage go under the name of naive set theory. After the discovery of paradoxes within naive set theory (such as Russell's paradox, Cantor's paradox and the Burali-Forti paradox), various axiomatic systems were proposed in the early twentieth century, of which ZermeloFraenkel set theory (with or without the axiom of choice) is still the best-known and most studied.Set theory is commonly employed as a foundational system for the whole of mathematics, particularly in the form of ZermeloFraenkel set theory with the axiom of choice. Besides its foundational role, set theory also provides the framework to develop a mathematical theory of infinity, and has various applications in computer science (such as in the theory of relational algebra), philosophy, formal semantics, and evolutionary dynamics. Its foundational appeal, together with its paradoxes, and its implications for the concept of infinity and its multiple applications have made set theory an area of major interest for logicians and philosophers of mathematics. Contemporary research into set theory covers a vast array of topics, ranging from the structure of the real number line to the study of the consistency of large cardinals.Porphyrian tree by Purchotius (1730), presenting Aristotle's CategoriesThe basic notion of grouping objects has existed since at least the emergence of numbers, and the notion of treating sets as their own objects has existed since at least the Tree of Porphyry in 3rd-century AD. The simplicity and ubiquity of sets makes it hard to determine the origin of sets as now used in mathematics; however, Bernard Bolzano's Paradoxes of the Infinite (Paradoxien des Unendlichen, 1851) is generally considered the first rigorous introduction of sets to mathematics. In his work, he (among other things) expanded on Galileo's paradox, and introduced one-to-one correspondence of infinite sets, for example between the intervals [0 , 5] {\displaystyle [0,5]} and [0 , 12] {\displaystyle [0,12]} by the relation 5 y = 12 x {\displaystyle 5y=12x} . However, he resisted saying these sets were equinumerous, and his work is generally considered to have been uninfluential in mathematics of his time.[1][2]Before mathematical set theory, basic concepts of infinity were considered to be in the domain of philosophy (see: Infinity (philosophy) and Infinity History). Since the 5th century BC, beginning with Greek philosopher Zeno of Elea in the West (and early Indian mathematicians in the East), mathematicians had struggled with the concept of infinity. With the development of calculus in the late 17th century, philosophers began to generally distinguish between potential and actual infinity, wherein mathematics was only considered in the latter.[3] Carl Friedrich Gauss famously stated: "Infinity is nothing more than a figure of speech which helps us talk about limits. The notion of a completed infinity doesn't belong in mathematics."[4]Development of mathematical set theory was motivated by several mathematicians. Bernhard Riemann's lecture On the Hypotheses which lie at the Foundations of Geometry (1854) proposed new ideas about topology. His lectures also introduced the concept of basing mathematics in terms of sets or manifolds in the sense of a class (which he called Mannigfaltigkeit) now called point-set topology. The lecture was published by Richard Dedekind in 1868, along with Riemann's paper on trigonometric series (which presented the Riemann integral), The latter was a starting point a movement in real analysis for the study of seriously discontinuous functions. A young Georg Cantor entered into this area, which led him to the study of point-sets. Around 1871, influenced by Riemann, Dedekind began working with sets in his publications, which dealt very clearly and precisely with equivalence relations, partitions of sets, and homomorphisms. Thus, many of the usual set-theoretic procedures of twentieth-century mathematics go back to his work. However, he did not publish a formal explanation of his set theory until 1888.Main article: Naive set theoryGeorg Cantor, 1894Set theory, as understood by modern mathematicians, is generally considered to be founded by a single paper in 1874 by Georg Cantor titled On a Property of the Collection of All Real Algebraic Numbers.[5][6][7] In his paper, he developed the notion of cardinality, comparing the sizes of two sets by setting them in one-to-one correspondence. His "revolutionary discovery" was that the set of all real numbers is uncountable, that is, one cannot put all real numbers in a list. This theorem is proved using Cantor's first uncountability proof, which differs from the more familiar proof using his diagonal argument.Cantor introduced fundamental constructions in set theory, such as the power set of a set A, which is the set of all possible subsets of A. He later proved that the size of the power set of A is strictly larger than the size of A, even when A is an infinite set; this result soon became known as Cantor's theorem. Cantor developed a theory of transfinite numbers, called cardinals and ordinals, which extended the arithmetic of the natural numbers. His notation for the cardinal numbers was the Hebrew letter {\displaystyle \aleph } (, aleph) with a natural number subscript; for the ordinals he employed the Greek letter {\displaystyle \omega } (, omega).Set theory was beginning to become an essential ingredient of the new modern approach to mathematics. Originally, Cantor's theory of transfinite numbers was regarded as counter-intuitive even shocking. This caused it to encounter resistance from mathematical contemporaries such as Leopold Kronecker and Henri Poincar and later from Hermann Weyl and L. E. J. Brouwer, while Ludwig Wittgenstein raised philosophical objections (see: Controversy over Cantor's theory).[a] Dedekind's algebraic style only began to find followers in the 1890sGottlob Frege, c.1879Despite the controversy, Cantor's set theory gained remarkable ground around the turn of the 20th century with the work of several notable mathematicians and philosophers. Richard Dedekind, around the same time, began working with sets in his publications, and famously constructing the real numbers using Dedekind cuts. He also worked with Giuseppe Peano in developing the Peano axioms, which formalized natural-number arithmetic, using set-theoretic ideas, which also introduced the epsilon symbol for set membership. Possibly most prominently, Gottlob Frege began to develop his Foundations of Arithmetic.In his work, Frege tries to ground all mathematics in terms of logical axioms using Cantor's cardinality. For example, the sentence "the number of horses in the barn is four" means that four objects fall under the concept horse in the barn. Frege attempted to explain our grasp of numbers through cardinality ('the number of...', or N x : F x {\displaystyle Nx:Fx}), relying on Hume's principle.Bertrand Russell, 1936However, Frege's work was short-lived, as it was found by Bertrand Russell that his axioms lead to a contradiction. Specifically, Frege's Basic Law V (now known as the axiom schema of unrestricted comprehension). According to Basic Law V, for any sufficiently well-defined property, there is the set of all and only the objects that have that property. The contradiction, called Russell's paradox, is shown as follows:Let R be the set of all sets that are not members of themselves. (This set is sometimes called "the Russell set".) If R is not a member of itself, then its definition entails that it is a member of itself; yet, if it is a member of itself, then it is not a member of itself, since it is the set of all sets that are not members of themselves. The resulting contradiction is Russell's paradox. In symbols: Let R = { x x x } , then R R R R {\displaystyle {\text{Let }}R=\{x\mid xot \in x\} {\text{, then }}R\in R\iff R \in R} This came around a time of several paradoxes or counter-intuitive results. For example, that the parallel postulate cannot be proved, the existence of mathematical objects that cannot be computed or explicitly described, and the existence of theorems of arithmetic that cannot be proved with Peano arithmetic. The result was a foundational crisis of mathematics.Main articles: Set (mathematics) and Algebra of setsSet theory begins with a fundamental binary relation between an object o and a set A. If o is a member (or element) of A, the notation o A is used. A set is described by listing elements separated by commas, or by a characterizing property of its elements, within braces { }.[8] Since sets are objects, the membership relation can relate sets as well, i.e., sets themselves can be members of other sets.A derived binary relation between two sets is the subset relation, also called set inclusion. If all the members of set A are also members of set B, then A is a subset of B, denoted A B. For example, {1, 2} is a subset of {1, 2, 3}, and so is {2} but {1, 4} is not. As implied by this definition, a set is a subset of itself. For cases where this possibility is unsuitable or would make sense to be rejected, the term proper subset is defined, variously denoted A B {\displaystyle A\subset B} , A B {\displaystyle A\subseteq B} , or A B {\displaystyle A\subseteq B} (note however that the notation A B {\displaystyle A\subset B} is sometimes used synonymously with A B {\displaystyle A\subseteq B} ; that is, allowing the possibility that A and B are equal). We call A a proper subset of B if and only if A is a subset of B, but A is not equal to B. Also, 1, 2, and 3 are members (elements) of the set {1, 2, 3}, but are not subsets of it; and in turn, the subsets, such as {1}, are not members of the set {1, 2, 3}. More complicated relations can exist; for example, the set {1} is both a member and a proper subset of the set {1, {1}}. Just as arithmetic features binary operations on numbers, set theory features binary operations on sets.[9] The following is a partial list of them:Union of the sets A and B, denoted A B, is the set of all objects that are a member of A, or B, or both.[10] For example, the union of {1, 2, 3} and {2, 3, 4} is the set {1, 2, 3, 4}.Intersection of the sets A and B, denoted A B, is the set of all objects that are members of both A and B.[11] For example, the intersection of {1, 2, 3} and {2, 3, 4} is the set {2, 3}.Set difference of U and A, denoted U A, is the set of all members of U that are not members of A.[12] The set difference {1, 2, 3} {2, 3, 4} is {1}, while conversely, the set difference {2, 3, 4} {1, 2, 3} is {4}. When A is a subset of U, the set difference U A is also called the complement of A in U. In this case, if the choice of U is clear from the context, the notation Ac is sometimes used instead of U A, particularly if U is a universal set as in the study of Venn diagrams.[13]Symmetric difference of sets A and B, denoted A B or A B, is the set of all objects that are a member of exactly one of A and B (elements which are in one of the sets, but not in both). For instance, for the sets {1, 2, 3} and {2, 3, 4}, the symmetric difference set is {1, 4}. It is the set difference of the union and the intersection, (A B) (A B) or (A B) (B A).Cartesian product of A and B, denoted A B, is the set whose members are all possible ordered pairs (a, b), where a is a member of A and b is a member of B. For example, the Cartesian product of {1, 2} and {red, white} is {(1, red), (1, white), (2, red), (2, white)}.[14]Some basic sets of central importance are the set of natural numbers, the set of real numbers and the empty set the unique set containing no elements. The empty set is also occasionally called the null set,[15] though this name is ambiguous and can lead to several interpretations. The empty set can be denoted with empty braces " { } {\displaystyle \{\}} " or the symbol " {\displaystyle \varnothing } " or " {\displaystyle \emptyset } ".The power set of a set A, denoted P (A) {\displaystyle {\mathcal {P}}(A)} , is the set whose members are all of the possible subsets of A. For example, the power set of {1, 2} is { {}, {1}, {2}, {1, 2} }. Notably, P (A) {\displaystyle {\mathcal {P}}(A)} contains both A and the empty set.Main article: von Neumann universeAn initial segment of the von Neumann hierarchyA set is pure if all of its members are sets, all members of its members are sets, and so on. For example, the set containing only the empty set is a nonempty pure set. In modern set theory, it is common to restrict attention to the von Neumann universe of pure sets, and many systems of axiomatic set theory are designed to axiomatize the pure sets only. There are many technical advantages to this restriction, and little generality is lost, because essentially all mathematical concepts can be modeled by pure sets. Sets in the von Neumann universe are organized into a cumulative hierarchy, based on how deeply their members, members of members, etc. are nested. Each set in this hierarchy is assigned (by transfinite recursion) an ordinal number {\displaystyle \alpha } , known as its rank. The rank of a pure set X {\displaystyle X} is defined to be the least ordinal that is strictly greater than the rank of any of its elements. For example, the empty set is assigned rank 0, while the set containing only the empty set is assigned rank 1. For each ordinal {\displaystyle \alpha } , the set V {\displaystyle V_{\alpha }} is defined to consist of all pure sets with rank less than {\displaystyle \alpha } . The entire von Neumann universe is denoted V {\displaystyle V} .Elementary set theory can be studied informally and intuitively, and so can be taught in primary schools using Venn diagrams. The intuitive approach tacitly assumes that a set may be formed from the class of all objects satisfying any particular defining condition. This assumption gives rise to paradoxes, the simplest and best known of which are Russell's paradox and the Burali-Forti paradox. Axiomatic set theory was originally devised to rid set theory of such paradoxes.[note 1]The most widely studied systems of axiomatic set theory imply that all sets form a cumulative hierarchy.[b] Such systems come in two flavors, those whose ontology consists of:Sets alone. This includes the most common axiomatic set theory, ZermeloFraenkel set theory with the axiom of choice (ZFC). Fragments of ZFC include:Zermelo set theory, which replaces the axiom schema of replacement with that of separation;General set theory, a small fragment of Zermelo set theory sufficient for the Peano axioms and finite sets;KripkePlatek set theory, which omits the axioms of infinity, powerset, and choice, and weakens the axiom schemata of separation and replacement.Sets and proper classes. These include Von NeumannBernaysGdel set theory, which has the same strength as ZFC for theorems about sets alone, and MorseKelley set theory and TarskiGrothendieck set theory, both of which are stronger than ZFC.The above systems can be modified to allow urelements, objects that can be members of sets but that are not themselves sets and do not have any members.The New Foundations systems of NFU (allowing urelements) and NF (lacking them), associate with Willard Van Orman Quine, are not based on a cumulative hierarchy. NF and NFU include a "set of everything", relative to which every set has a complement. In these systems urelements matter, because NF, but not NFU, produces sets for which the axiom of choice does not hold. Despite NF's ontology not reflecting the traditional cumulative hierarchy and violating well-foundedness, Thomas Forster has argued that it does reflect an iterative conception of set.[16]Systems of constructive set theory, such as CST, CZF, and IZF, embed their set axioms in intuitionistic instead of classical logic. Yet other systems accept classical logic but feature a nonstandard membership relation. These include rough set theory and fuzzy set theory, in which the value of an atomic formula embodying the membership relation is not simply True or False. The Boolean-valued models of ZFC are a related subject.An enrichment of ZFC called internal set theory was proposed by Edward Nelson in 1977.[17]Many mathematical concepts can be defined precisely using only set theoretic concepts. For example, mathematical structures as diverse as graphs, manifolds, rings, vector spaces, and relational algebras can all be defined as sets satisfying various (axiomatic) properties. Equivalence and order relations are ubiquitous in mathematics, and the theory of mathematical relations can be described in set theory.[18][19]Set theory is also a promising foundational system for much of mathematics. Since the publication of the first volume of Principia Mathematica, it has been claimed that most (or even all) mathematical theorems can be derived using an aptly designed set of axioms for set theory, augmented with many definitions, using first or second-order logic. For example, properties of the natural and real numbers can be derived within set theory, as each of these number systems can be defined by representing their elements as sets of specific forms.[20]Set theory as a foundation for mathematical analysis, topology, abstract algebra, and discrete mathematics is likewise uncontroversial; mathematicians accept (in principle) that theorems in these areas can be derived from the relevant definitions and the axioms of set theory. However, it remains that few full derivations of complex mathematical theorems from set theory have been formally verified, since such formal derivations are often much longer than the natural language proofs mathematicians commonly present. One verification project, Metamath, includes human-written, computer-verified derivations of more than 12,000 theorems starting from ZFC set theory, first-order logic and propositional logic.[21]Set theory is a major area of research in mathematics with many interrelated subfields:Main article: Infinitary combinatoricsCombinatorial set theory concerns extensions of finite combinatorics to infinite sets. This includes the study of cardinal arithmetic and the study of extensions of Ramsey's theorem such as the ErdsRado theorem.Main article: Descriptive set theoryDescriptive set theory is the study of subsets of the real line and, more generally, subsets of Polish spaces. It begins with the study of pointclasses in the Borel hierarchy and extends to the study of more complex hierarchies such as the projective hierarchy and the Wadge hierarchy. Many properties of Borel sets can be established in ZFC, but proving these properties hold for more complicated sets requires additional axioms related to determinacy and large cardinals.The field of effective descriptive set theory is between set theory and recursion theory. It includes the study of lightface pointclasses, and is closely related to hyperarithmetical theory. In many cases, results of classical descriptive set theory have effective versions; in some cases, new results are obtained by proving the effective version first and then extending ("relativizing") it to make it more broadly applicable.A recent area of research concerns Borel equivalence relations and more complicated definable equivalence relations. This has important applications to the study of invariants in many fields of mathematics.Main article: Fuzzy set theoryIn set theory as Cantor defined and Zermelo and Fraenkel axiomatized, an object is either a member of a set or not. In fuzzy set theory this condition was relaxed by Lotfi A. Zadeh so an object has a degree of membership in a set, a number between 0 and 1. For example, the degree of membership of a person in the set of "tall people" is more flexible than a simple yes or no answer and can be a real number such as 0.75.Main article: Inner model theoryAn inner model of ZermeloFraenkel set theory (ZF) is a transitive class that includes all the ordinals and satisfies all the axioms of ZF. The canonical example is the constructible universe L developed by Gdel.One reason that the study of inner models is of interest is that it can be used to prove consistency results. For example, it can be shown that regardless of whether a model V of ZF satisfies the continuum hypothesis or the axiom of choice, the inner model L constructed inside the original model will satisfy both the generalized continuum hypothesis and the axiom of choice. Thus the assumption that ZF is consistent (has at least one model) implies that ZF together with these two principles is consistent.The study of inner models is common in the study of determinacy and large cardinals, especially when considering axioms such as the axiom of determinacy that contradict the axiom of choice. Even if a fixed model of set theory satisfies the axiom of choice, it is possible for an inner model to fail to satisfy the axiom of choice. For example, the existence of sufficiently large cardinals implies that there is an inner model satisfying the axiom of determinacy (and thus not satisfying the axiom of choice).[22]Main article: Large cardinal propertyA large cardinal is a cardinal number with an extra property. Many such properties are studied, including inaccessible cardinals, measurable cardinals, and many more. These properties typically imply the cardinal number must be very large, with the existence of a cardinal with the specified property unprovable in ZermeloFraenkel set theory.Main article: DeterminacyDeterminacy refers to the fact that, under appropriate assumptions, certain two-player games of perfect information are determined from the start in the sense that one player must have a winning strategy. The existence of these strategies has important consequences in descriptive set theory, as the assumption that a broader class of games is determined often implies that a broader class of sets will have a topological property. The axiom of determinacy (AD) is an important object of study; although incompatible with the axiom of choice, AD implies that all subsets of the real line are well behaved (in particular, measurable and with the perfect set property). AD can be used to prove that the Wadge degrees have an elegant structure.Main article: Forcing (mathematics)Paul Cohen invented the method of forcing while searching for a model of ZFC in which the continuum hypothesis fails, or a model of ZF in which the axiom of choice fails. Forcing adjoins to some given model of set theory additional sets in order to create a larger model with properties determined (i.e. "forced") by the construction and the original model. For example, Cohen's construction adjoins additional subsets of the natural numbers without changing any of the cardinal numbers of the original model. Forcing is also one of two methods for proving relative consistency by finitistic methods, the other method being Boolean-valued models.Main article: Cardinal characteristics of the continuumA cardinal invariant is a property of the real line measured by a cardinal number. For example, a well-studied invariant is the smallest cardinality of a collection of meagre sets of reals whose union is the entire real line. These are invariants in the sense that any two isomorphic models of set theory must give the same cardinal for each invariant. Many cardinal invariants have been studied, and the relationships between them are often complex and related to axioms of set theory.Main article: Set-theoretic topologySet-theoretic topology studies questions of general topology that are set-theoretic in nature or that require advanced methods of set theory for their solution. Many of these theorems are independent of ZFC, requiring stronger axioms for their proof. A famous problem is the normal Moore space question, a question in general topology that was the subject of intense research. The answer to the normal Moore space question was eventually proved to be independent of ZFC.Main article: Controversy over Cantor's theoryFrom set theory's inception, some mathematicians have objected to it as a foundation for mathematics. The most common objection to set theory, one Kronecker voiced in set theory's earliest years,

disjoint if AB = . Three or more sets are said to be disjoint if every two of them are disjoint. The notation A-B is used to indicate the set of all elements of A that are not elements of B. This operation has no standard name, but when B is a subset of A, A-B is sometimes said to be the complement of B in A. The relationships among sets are often represented pictorially by a Venn diagram, in which sets are represented as the interiors of overlapping circles (or other plane figures). Set combinations are represented by areas bounded by the circles, as shown in the following example for two sets: 2. Ordered Pairs An ordered pair is a set of two elements in a specified order. An ordered pair is usually written (a,b) where a is the first element and b is the second element. Two ordered pairs (a,b) and (c,d) are equal if a=c and b=d. Reversing the elements of an ordered pair produces a different ordered pair if the elements are not the same. For example, the ordered pair (1,2) is not equal to the ordered pair (2,1). For two sets A and B, the cross product A B is the set of all ordered pairs whose first and second elements are elements of A and B, respectively. That is, A B = {(a,b) a A and b B} Ordered triples, quadruples, etc. could be defined, but they are seldom needed. 3. Relations A relation R on a set A is simply a set of ordered pairs of elements of A, i.e., R A A. Two elements a and b are said to obey the relation if (a,b) is in R. However, for most relations, the set notation is not used. Instead, a symbol such as ~ placed between the elements to indicate that they obey the relation; for example a~b means that (a,b) is in R. Other symbols often used for relations are = > < Most useful relations have some additional properties. A relation ~ on the set A is an equivalence if the following hold for every a, b and c in A: It is reflexive: a~a. It is symmetric: a~b implies that b~a. It is transitive: a~b and b~c imply that a~c. A set of nonempty subsets of a set A is called a partition of A if each element of A belongs to one and only one of the subsets; i.e., if the subsets are disjoint and their union is A. The following theorem establishes a connection between an equivalence relation and a partition. Theorem 3.1. If ~ is an equivalence relation on the set A, then there is partition of the set A such that a~b if, and only if, a and b belong to the same set in the partition. Conversely, if P is a partition of A, then "a and b belong to the same set in P" is an equivalence relation. Proof. Consider the set P of subsets Ta = {x A | x~a}. Clearly every a in A belongs to at least one subset in P, namely Ta. Hence the sets in P are nonempty and their union is A. Now let Ta and Tb be two subsets in P. If they have an element c in common, then c~a, c~b and x~b for every x Tb. By transitivity x~a and x Ta, too. Similar arguments show that every element of Ta is also an element of Tb. Hence Ta and Tb are equal. If two subsets in P have no element in common, they are disjoint. Hence P is the desired partition. The converse is trivial. The sets in the partition associated in this way with an equivalence relation are called its equivalence classes. They are often used to define mathematical systems. Equivalence relations on two sets A and B can be used to define an equivalence relation on A B in the obvious way: (a,b) is equivalent to (c,d) if a is equivalent to c and b is equivalent to d. 4. Order A partial order on a set A is a relation with the following properties for every a, b and c in A: It is reflexive: a a. It is antisymmetric: a b and b a imply that a = b. It is transitive: a b and b c imply that a c. A partial order on the set A is called a linear order (or a total order) if, for every two elements a and b of A, a b or b a (or both, if a = b). The set of all subsets of a set is partially ordered by inclusion: S T means S T. This partial order is usually not a total order because we can find two subsets, such as {1,2,3} and {2,3,4}, such that neither is a subset of the other. The familiar relation in arithmetic is a total order. In working with a partial or total order, it is common to define some associated relations: a b means b a, a < b means a b and a b, a > b means b a and b a. There is an alternative way to define partial and total orders. A relation < is a partial order if obeys the following two conditions: It is transitive: a < b and b < c imply that a < c. a < a is always false. A partial order is a total order if it is also trichotomous: for any two elements a and b, one and only one of the following holds: The other relations are then defined in terms of b means b < a. It can be shown that the two ways of defining partial and total orders are equivalent. Generally, the names "partial order" and "total order" are applied to the entire set of relations , and without specifying which is the order relation and which are associated with it. 5. Functions A function f from the set A to the set B is a rule which, given any element x of A, produces exactly one corresponding element of B represented by f(x). This concept is often expressed symbolically as f:AB. A function is also called a mapping. Both names are commonly used in mathematics, but from this point forth we will use the name function. A second, and more abstract, way to define a function f:AB is as subset of A B such that for every element x of A there is one and only one ordered pair in the subset whose first element is x. The second element of the pair is then defined to be f(x). The element f(x) is called the image of x under the function. The function f is also said to map or carry the element x to the element f(x). If f:AB then A is called the domain of f, and the set of all elements of B which are images of elements in A is called the range of f. The sets A and B need not be different; in fact, they are the same in many applications. Some functions have special properties that make them especially interesting or useful. If f:AB, then If the range is equal to B, then f is called a surjection, a surjective function, or a function of A onto B. The function is sometimes also said to be "onto", but the use of a preposition as an adjective sounds so stilted that good writers tend to avoid it. If the range is a proper subset of B, then f is called a function of A into B. If the f carries at most one element of its domain into each element of its range, i.e., if f(x) = f(y) implies that x = y, then f is called an injection, an injective function, or a one-to-one function. If f is both surjective and one-to-one then it is called a one-to-one correspondence of A and B. If f is one-to-one but not surjective, then it is a one-to-one correspondence of A and its range, which is a proper subset of B. If f:AB is a one-to-one correspondence then it has an inverse function called f -1:BA defined by f -1(x) = the element w of A such that f(w) = x. Of course, the converse is also true. If a function has an inverse, then it is a one-to-one correspondence. Two functions f:AB and g:AB are equal if f(x) = g(x) for every x in A. If the range of one function is a subset of the domain of another, then a composite function is defined by applying the functions successively. That is, if f:AB and g:BC then the composite function (fg):AC is defined by (f g)(x) = g(f(x)) for every x in A. If the functions have appropriate domains and ranges, composition is associative, i.e, (f g) h = f (g h). A one-to-one function f:AB of two sets with some structure is called an isomorphism if it preserves the structure. We have seen one example of a set with structure: the partially ordered set. If the sets A and B are partially ordered, then f:AB is an isomorphism if it is one-to-one, surjective, and f(x) < f(y) if, and only if, x < y. An isomorphism of a structured set with itself is called an automorphism. Clearly, the identity function (f(x) = x for all x) is an automorphism of any structured set. A good example of a nontrivial automorphism is the function which carries a complex number into its conjugate (i.e,. f(x+iy) = x-iy for all real x and y). It is one-to-one, surjective, and preserves addition and multiplication of complex numbers. Suppose f:AB and there are equivalence relations on the sets A and B. Let PA and PB be the corresponding sets of equivalence classes of A and B, respectively. If f carries equivalent elements of A into equivalent elements of B, i.e., if a ~ b implies that f(a) ~ f(b), then there is a unique function g:PA PB defined in the following manner. Let S be an equivalence class in PA. Select any element a from this equivalence class and define g(S) to be the equivalence class containing f(a). It is easily shown that this does not depend on the particular element chosen from PA. Moreover, g inherits many of the properties of f; e.g., if f is surjective, so is g. 5. Operations A unary operation on a set is a function whose domain is that set. What distinguishes a unary operation from an ordinary function is the notation used, and often its relationships with other functions or operations. For example, the function that carries any real number x to the number -x is a unary operation called negation. The range of the function is often the same set, but this is not required. A binary operation is a function whose domain is the cross product of two sets (or the cross product of a set with itself). For example, addition and multiplication are two binary operations on the set R R, where R is the set of real numbers. The image of an ordered pair (x,y) is usually written as x+y for addition and xy for multiplication. Here x and y are called the operands. The former notation is usually used only for addition, or operations very much like addition. The latter notation is used for more general operations. Unary and binary operations are very common in mathematics; operations with three or more operands are rare, except for extensions of binary operations as noted below. Binary operations on a single set are more common than binary operations on pairs of sets, but both are encountered frequently. A binary operation is said to be associative if it can be used on three operands without regard to their grouping, i.e., if (xy)z = x(yz) (x + y) + z = x + (y + z) If a binary operation is associative, we can write the result for three operands without parentheses, making it a well-defined operation with three operands: xyz = (xy)z = x(yz) x + y + z = (x + y) + z = x + (y + z) Ordinary addition and multiplication are associative; so are many other binary operations. The composition of functions is an associative binary operation (provided the functions have suitable domains and ranges). In fact, most binary operations are associative. If a binary operation is associative, it is easy to extend the property to operations on four or more operands: wxyz = (wxy)z = (w(xy))z = w((xy)z) = w(xyz). A binary operation is commutative if the order of the operands does not affect the result; i.e., if xy = yx x + y = y + x If a commutative operation is also associative, commutativity can easily be extended to operations on three or more operands: xyz = (xy)z = (yx)z = yxz = y(xz) = (yx)z = yxz, etc. Binary operations which are commutative but not associative are very rare. Binary operations which are associative but not commutative are fairly common. Composition of functions is one example; a demonstration of that fact will be given later. Now consider a binary operation on A B with its range in C (which need not be different sets). Suppose there are equivalence operations on these sets (which also need not be different), and the binary operation preserves equivalence, i.e., the operation when applied to equivalent operands gives equivalent results, or a ~ b and c ~ d imply that ac ~ bd. Then, just as a function of one variable was extended to a function on equivalence classes, an operation on two variables can be extended similarly. If a and b are any elements of the equivalence classes P and Q, respectively, then PQ is defined to be the equivalence class containing ab. The new operation inherits many properties of the old one, including associativity and commutativity. When dealing with set theory, there are a number of operations to make new sets out of old ones. One of the most common set operations is called the intersection. Simply stated, the intersection of two sets A and B is the set of all elements that both A and B have in common. We will look at details concerning the intersection in set theory. As we will see, the key word here is the word "and." For an example of how the intersection of two sets forms a new set, lets consider the sets A = {1, 2, 3, 4, 5} and B = {3, 4, 5, 6, 7, 8}. To find the intersection of these two sets, we need to find out what elements they have in common. The numbers 3, 4, 5 are elements of both sets, therefore the intersections of A and B is {3. 4. 5]. In addition to understanding the concepts concerning set theory operations, it is important to be able to read symbols used to denote these operations. The symbol for intersection is sometimes replaced by the word and between two sets. This word suggests the more compact notation for an intersection that is typically used. The symbol used for the intersection of the two sets A and B is given by A B. One way to remember that this symbol refers to intersection is to notice its resemblance to a capital A, which is short for the word "and." To see this notation in action, refer back the above example. Here we had the sets A = {1, 2, 3, 4, 5} and B = {3, 4, 5, 6, 7, 8}. So we would write the set equation A B = {3, 4, 5}. One basic identity that involves the intersection shows us what happens when we take the intersection of any set with the empty set, denoted by #8709. The empty set is the set with no elements. If there are no elements in at least one of the sets we are trying to find the intersection of, then the two sets have no elements in common. In other words, the intersection of any set with the empty set will give us the empty set. This identity becomes even more compact with the use of our notation. We have the identity: A = . For the other extreme, what happens when we examine the intersection of a set with the universal set? Similar to how the word universe is used in astronomy to mean everything, the universal set contains every element. It follows that every element of our set is also an element of the universal set. Thus the intersection of any set with the universal set is the set that we started with. Again our notation comes to the rescue to express this identity more succinctly. For any set A and the universal set U, A U = A. There are many more set equations that involve the use of the intersection operation. Of course, it's always good to practice using the language of set theory. For all sets A, and B and D we have:Reflexive Property: A A =ACommutative Property: A B = B AAssociative Property: (A B) D =A (B D)Distributive Property: (A B) D = (A D) (B D)DeMorgans Law I: (A B)C = AC BCDeMorgans Law II: (A B)C = AC BC Share copy and redistribute the material in any medium or format for any purpose, even commercially. Adapt remix, transform, and build upon the material for any purpose, even commercially. The licensor cannot revoke these freedoms as long as you follow the license terms. Attribution You must give appropriate credit , provide a link to the license, and indicate if changes were made . You may do so in any reasonable manner, but not in any way that suggests the licensor endorses you or your use. ShareAlike If you remix, transform, or build upon the material, you must distribute your contributions under the same license as the original. No additional restrictions You may not apply legal terms or technological measures that legally restrict others from doing anything the license permits. You do not have to comply with the license for elements of the material in the public domain or where your use is permitted by an applicable exception or limitation . No warranties are given. The license may not give you all of the permissions necessary for your intended use. For example, other rights such as publicity, privacy, or moral rights may limit how you use the material.

Elementary set theory pdf. Elementary set theory in real analysis. What is elementary set. What is elements in set theory. What is elementary counting in set theory. What is set theory. What is set theory with examples.

- sears lawn mower model 944
- self propelled lawn mower vs walk behind
- https://ratioabroadstudies.com/ckfinder/userfiles/files/0c587b55-7d36-4940-829a-43a6adc4e203.pdf
- http://qianshunqs.com/admin/upload/21083792479.pdf
- https://bdprescription.com/ci/userfiles/files/08d207d1-c78b-458f-9450-29a18fb64736.pdf
- gavaka
- http://oipipleszno.pl/userfiles/file/9cc846a3-ec3c-4f4a-a2f8-0deef285ac4d.pdf